AUGUST, 1886.

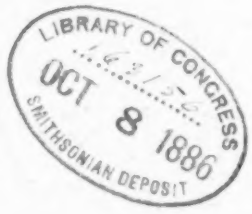
ANNALS OF MATHEMATICS.

EDITED BY

ORMOND STONE AND WILLIAM M. THORNTON.

OFFICE OF PUBLICATION: UNIVERSITY OF VIRGINIA.

Volume 2, Number 4.



CHARLOTTESVILLE, VA.:
Printed for the Editors by BLAKEY & PROUT, Steam Book and Job Printers.

Agents: B. WESTERMANN & Co., New York.

ANNALS OF MATHEMATICS.

VOL. II. AUGUST, 1886. NO. 4.

ON THE USE OF SOMOFF'S THEOREM FOR THE EVALUATION OF THE ELLIPTIC INTEGRAL OF THE THIRD SPECIES.

By MR. CHAS. H. KUMMELL, Washington, D. C.

The theorem which I call Somoff's, is a relation between elliptic integrals of the third species in successive steps in the scale of moduli and corresponding amplitudes and parametric angles. I had, however, independently discovered this theorem, and founded on it the methods which I am going to explain, before I saw Somoff's article in *Crelle's Journal*, Vol. XLVII, pp. 269–289, and since they are essentially different from Somoff's methods, and in some respects more convenient, it may be of some use to give them also.

If, according to the notation which I have used in previous articles,

$$\varphi_\gamma = \int_0^\varphi \frac{d\varphi}{\sqrt{(1-\gamma^2\sin^2\varphi)}} = \int_0^\varphi \frac{d\varphi}{\sqrt{(\beta^2+\gamma^2\cos^2\varphi)}} = \int_0^\varphi \frac{d\varphi}{\Delta\varphi}, \tag{1}$$

then Jacobi's type form of the elliptic integral of the third species is

$$\begin{aligned} \Pi(\varphi_\gamma, \mu_\gamma) &= \int_0^\varphi \frac{d\varphi}{\Delta\varphi} \cdot \frac{\gamma^2\sin^2\varphi}{1-\gamma^2\sin^2\mu\sin^2\varphi} \sin\mu\cos\mu\,\Delta\mu \\ &= \int_0^\varphi \frac{d\varphi}{\Delta\varphi} \cdot \frac{1-\Delta^2\varphi}{\cos^2\mu+\sin^2\mu\,\Delta^2\varphi} \sin\mu\cos\mu\,\Delta\mu \\ &= \frac{\Delta\mu}{\tan\mu}[\Pi_1(\varphi,\gamma,-\gamma^2\sin^2\mu)-\varphi_\gamma], \end{aligned} \tag{2}$$

where Π_1 denotes Legendre's type for the third species, in which the parameter

$$n = -\gamma^2\sin^2\mu. \tag{3}$$

This integral vanishes for all values of μ which cause either $\sin\mu$, $\cos\mu$, or $\Delta\mu$

to vanish; viz.: if $\mu = 2m\,\Delta$,* or $= (2m'+1)\,\Delta$, or $= [(2m+1)\Delta_\gamma + (2m'+1)\,\Delta_\beta i]_{-\gamma}$, or, in the simplest cases, if $\mu = 0$, or $= \Delta$, or $= (\Delta_\gamma + \Delta_\beta i)_{-\gamma}$.

The parametric angle μ cannot be real in all cases, and it is necessary to assume the more general form,

$$n = -\gamma^2 \sin^2(\mu_\gamma + \nu_\beta i)_{-\gamma}$$

$$= -\left(\frac{\gamma}{1 - \Delta^2\mu \sin^2\nu}\right)^2 [\sin\mu\,\Delta(\nu_\beta)_{-\beta} + i\cos\mu\,\Delta\mu\sin\nu\cos\nu]^2. \quad (4)$$

This is real if

1. $\sin\nu = 0$, then $n = -\gamma^2\sin^2\mu$; $\quad (0 < -n < \gamma^2)$ (4_1)

2. $\sin\mu = 0$, then $n = -\gamma^2\sin^2(\nu_\beta i)_{-\gamma} = \gamma^2\tan^2\nu$; $\quad (-\infty < -n < 0)$ (4_2)

3. $\cos\nu = 0$, then $n = -\gamma^2\sin^2(\mu_\gamma + \Delta_\beta i)_{-\gamma} = -\dfrac{1}{\sin^2\mu}$; $\quad (1 < -n < \infty)$ (4_3)

4. $\cos\mu = 0$, then $n = -\gamma^2\sin^2(\Delta_\gamma + \nu_\beta i)_{-\gamma} = -\dfrac{\gamma^2}{\Delta^2(\nu_\beta)_{-\beta}}$. $\quad (\gamma^2 < -n < 1)$ (4_4)

This completes the range of real values for parameter. There are, however, four other forms extending over the same ranges respectively, which are obtained from these by taking elliptic complements to μ or ν; thus,

$1'.\ n = -\gamma^2\sin^2(\Delta_\gamma + \mu_\gamma)_{-\gamma} = -\gamma^2\dfrac{\cos^2\mu}{\Delta^2\mu}, \quad (0 < -n < \gamma^2)$ $(4_1')$

$2'.\ n = -\gamma^2\sin^2(\Delta_\beta i + \nu_\beta i)_{-\gamma} = \gamma^2\cot^2\nu, \quad (-\infty < -n < 0)$ $(4_2')$

$3'.\ n = -\gamma^2\sin^2(\Delta_\gamma + \mu_\gamma + \Delta_\beta i)_{-\gamma} = -\dfrac{\Delta^2\mu}{\cos^2\mu}, \quad (1 < -n < \infty)$ $(4_3')$

$4'.\ n = -\gamma^2\sin^2(\Delta_\gamma + \Delta_\beta i + \nu_\beta i)_{-\gamma} = -\Delta^2(\nu_\beta)_{-\beta}. \quad (\gamma^2 < -n < 1)$ $(4_4')$

These last four cases, which I give only for reference, are not essentially different from the first four, and they may replace each other respectively. By means of the theorem for the addition of parameters,

$$\Pi(\varphi_\gamma, \mu_\gamma + \nu_\beta i) = \Pi(\varphi_\gamma, \mu_\gamma) + \Pi(\varphi_\gamma, \nu_\beta i)$$
$$- \gamma^2\varphi_\gamma \sin\mu \sin(\nu_\beta i)_{-\gamma} \sin(\mu_\gamma + \nu_\beta i)_{-\gamma}$$
$$+ \tfrac{1}{2}l\frac{1 + \gamma^2\sin\mu\sin(\nu_\beta i)_{-\gamma}\sin\varphi\sin(\mu_\gamma + \nu_\beta i + \varphi_\gamma)_{-\gamma}}{1 - \gamma^2\sin\mu\sin(\nu_\beta i)_{-\gamma}\sin\varphi\sin(\mu_\gamma + \nu_\beta i - \varphi_\gamma)_{-\gamma}}, \quad (5)$$

*See ANNALS OF MATHEMATICS, Vol. II, p. 38.

we can reduce all cases to the first two forms, viz.: —

1. the logarithmic integral,

$$\Pi(\varphi_\gamma, \mu_\gamma) = \int_0^\varphi \sin\mu\cos\mu\,\Delta\mu\,\frac{d\varphi}{\Delta\varphi}\cdot\frac{\gamma^2\sin^2\varphi}{1-\gamma^2\sin^2\mu\sin^2\varphi},$$

2. the cyclometric integral,

$$\Pi(\varphi_\gamma, \nu_\beta i) = i\int_0^\varphi \frac{\tan\nu\,\Delta(\nu_\beta)_{-\beta}}{\cos^2\nu}\cdot\frac{d\varphi}{\Delta\varphi}\cdot\frac{\gamma^2\sin^2\varphi}{1+\gamma^2\tan^2\nu\sin^2\varphi};$$

and we have the following special formulæ of reduction: —

$$\begin{aligned}\Pi(\varphi_\gamma, \lrcorner_\gamma + \mu_\gamma) &= \Pi(\varphi_\gamma, \mu_\gamma) - \gamma^2\varphi_\gamma\sin\mu\sin(\lrcorner_\gamma + \mu_\gamma)_{-\gamma} \\ &\quad + \tfrac{1}{2}l\frac{1+\gamma^2\sin\mu\sin\varphi\sin(\lrcorner_\gamma + \mu_\gamma + \varphi_\gamma)_{-\gamma}}{1-\gamma^2\sin\mu\sin\varphi\sin(\lrcorner_\gamma + \mu_\gamma - \varphi_\gamma)_{-\gamma}} \\ &= \Pi(\varphi_\gamma, \mu_\gamma) - \gamma^2\varphi_\gamma\sin\mu\frac{\cos\mu}{\Delta\mu} + \tfrac{1}{2}l\frac{\Delta(\mu_\gamma - \varphi_\gamma)_{-\gamma}}{\Delta(\mu_\gamma + \varphi_\gamma)_{-\gamma}}, \quad (6)\end{aligned}$$

$$\begin{aligned}\Pi(\varphi_\gamma, \lrcorner_\gamma + \nu_\beta i) &= \Pi(\varphi_\gamma, \nu_\beta i) - \gamma^2 i\varphi_\gamma\frac{\tan\nu}{\Delta(\nu_\beta)_{-\beta}} \\ &\quad + \tfrac{1}{2}l\frac{\Delta\varphi\cos\nu\,\Delta(\nu_\beta)_{-\beta} + i\sin\varphi\cos\varphi\sin\nu}{\Delta\varphi\cos\nu\,\Delta(\nu_\beta)_{-\beta} - i\sin\varphi\cos\varphi\sin\nu} \\ &= \Pi(\varphi_\gamma, \nu_\beta i) - \gamma^2 i\varphi_\gamma\frac{\tan\nu}{\Delta(\nu_\beta)_{-\beta}} \\ &\quad + i\,\text{arc tan}\left(\frac{\sin\varphi\cos\varphi}{\Delta\varphi}\cdot\frac{\tan\nu}{\Delta(\nu_\beta)_{-\beta}}\right), \quad (7)\end{aligned}$$

$$\Pi(\varphi_\gamma, \lrcorner_\gamma + \mu_\gamma + \lrcorner_\beta i) = \Pi(\varphi_\gamma, \mu_\gamma) - \varphi_\gamma\tan\mu\,\Delta\mu + \tfrac{1}{2}l\frac{\cos(\mu_\gamma - \varphi_\gamma)_{-\gamma}}{\cos(\mu_\gamma + \varphi_\gamma)_{-\gamma}}, \quad (8)$$

$$\begin{aligned}\Pi(\varphi_\gamma, \lrcorner_\gamma + \lrcorner_\beta i + \nu_\beta i) &= \Pi(\varphi_\gamma, \nu_\beta i) - \varphi_\gamma i\tan\nu\,\Delta(\nu_\beta)_{-\beta} \\ &\quad + i\,\text{arc tan}[\tan\varphi\Delta\varphi\tan\nu\,\Delta(\nu_\beta)_{-\beta}]. \quad (9)\end{aligned}$$

These formulæ suffice for the required reduction, and we may suppose given the simple logarithmic and cyclometric integral for evaluation.

Somoff's theorem I deduce as follows: —

Assume by Landen's transformation, ascending,

$$\sin\varphi = (1+\beta')\sin\varphi'\frac{\cos\varphi'}{\Delta\varphi'} = (1+\beta')\sin\varphi'\sin(\lrcorner_{\gamma'} - \varphi'_{\gamma'})_{-\gamma'}, \quad (10)$$

$$\sin\mu = (1+\beta')\sin\mu'\frac{\cos\mu'}{\Delta\mu'} = (1+\beta')\sin\mu'\sin(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}, \quad (11)$$

where $\beta', \gamma', \varphi', \mu'$ correspond to $\beta, \gamma, \varphi, \mu$ respectively on the next higher step of the modular scale. We have also

$$\gamma = \frac{1-\beta'}{1+\beta'}, \tag{12}$$

$$\Delta\mu = \frac{\Delta\mu' + \frac{\beta'}{\Delta\mu'}}{1+\beta'} = \frac{\Delta\mu' + \Delta(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}}{1+\beta'}, \tag{13}$$

$$\cos\mu = \frac{\Delta\mu' - \frac{\beta'}{\Delta\mu'}}{1-\beta'} = \frac{\Delta\mu' - \Delta(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}}{1-\beta'}, \tag{14}$$

$$\frac{d\varphi}{\Delta\varphi} = (1+\beta')\frac{d\varphi'}{\Delta\varphi'}; \tag{15}$$

whence

$$\Pi(\varphi_\gamma, \mu_\gamma) = \int_0^{\varphi'} (1+\beta')\frac{d\varphi'}{\Delta\varphi'}(1+\beta')\sin\mu'\frac{\cos\mu'}{\Delta\mu'}\cdot\frac{\Delta^2\mu' - \frac{\beta'^2}{\Delta^2\mu'}}{1-\beta'^2}\cdot\frac{(1-\beta')^2\sin^2\varphi'\frac{\cos^2\varphi'}{\Delta^2\varphi'}}{1-\gamma'^4\sin^2\mu'\frac{\cos^2\mu'}{\Delta^2\mu'}\sin^2\varphi'\frac{\cos^2\varphi'}{\Delta^2\varphi'}}$$

$$= \int_0^{\varphi'} \frac{d\varphi'}{\Delta\varphi'}\gamma'^2\sin^2\varphi'\sin\mu'\cos\mu'\Delta\mu'\cdot\frac{\left(\Delta^2\mu' - \frac{\beta'^2}{\Delta^2\mu'}\right)\cos^2\varphi'}{\Delta^2\mu'\Delta^2\varphi' - \gamma'^4\sin^2\mu'\sin^2\varphi'\cos^2\mu'\cos^2\varphi'}$$

$$= \int_0^{\varphi'} \frac{d\varphi'}{\Delta\varphi'}\gamma'^2\sin^2\varphi'\sin\mu'\cos\mu'\Delta\mu'\cdot\left\{\frac{1}{1-\gamma'^2\sin^2\mu'\sin^2\varphi'} - \frac{\frac{\beta'^2}{\Delta^4\mu'}}{1-\gamma'^2\frac{\cos^2\mu'}{\Delta^2\mu'}\sin^2\varphi'}\right\}.$$

Since
$$\sin(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'} = \frac{\cos\mu'}{\Delta\mu'},$$

$$\cos(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'} = \frac{\beta'\sin\mu'}{\Delta\mu'},$$

$$\Delta(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'} = \frac{\beta'}{\Delta\mu'};$$

we have
$$\sin(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}\cos(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}\Delta(\lrcorner_{\gamma'} - \mu'_{\gamma'})_{-\gamma'}$$
$$= \frac{\beta'^2\sin\mu'\cos\mu'}{\Delta^3\mu'},$$

and, therefore,

$$\Pi(\varphi\gamma, \mu\gamma) = \Pi(\varphi'\gamma', \mu'\gamma') - \Pi(\varphi'\gamma', \lrcorner\gamma' - \mu'\gamma')$$
$$= \Pi(\varphi'\gamma', \mu'\gamma') + \Pi(\varphi'\gamma', \lrcorner\gamma' + \mu'\gamma'), \qquad (16)$$

or, writing for brevity μ' for the co-amplitude of μ',

$$\Pi(\varphi\gamma, \mu\gamma) = \Pi(\varphi'\gamma', \mu'\gamma') - \Pi(\varphi'\gamma', \mu'\gamma'); \quad \mu'\gamma' + \mu'\gamma' = \lrcorner\gamma'. \qquad (16')$$

This I call Somoff's theorem. Somoff, however, makes no use of it in this form. He employs a relation which is obtained from (16) by expressing $\Pi(\varphi'\gamma', \lrcorner\gamma' - \mu'\gamma')$ in terms of $\Pi(\varphi'\gamma', \mu'\gamma')$ by (6). We have thus

$$\Pi(\varphi\gamma, \mu\gamma) = 2\Pi(\varphi'\gamma', \mu'\gamma') - \gamma'^2\varphi'\gamma' \sin\mu' \frac{\cos\mu'}{\lrcorner\mu'} + \tfrac{1}{2}l \frac{\lrcorner(\mu'\gamma' - \varphi'\gamma') - \gamma'}{\lrcorner(\mu'\gamma' + \varphi'\gamma') - \gamma'}$$

$$= 2\Pi(\varphi'\gamma', \mu'\gamma') - \gamma\varphi\gamma \sin\mu + \tfrac{1}{2}l \frac{1 + \gamma \sin\mu \sin\varphi}{1 - \gamma \sin\mu \sin\varphi}. \qquad (17)$$

[TO BE CONTINUED.]

A SIMPLE DISCUSSION OF LOGARITHMIC ERRORS.

By PROF. H. A. HOWE, Denver, Col.

[CONTINUED FROM VOL. II, PAGE 43.]

PROBLEM XI.

To find the probable error of an interpolated logarithmic trigonometric function.

Tables of logarithmic trigonometric functions may be divided into three classes. The first class embraces those in which the tabular difference is to be multiplied by a decimal of one, two, three, or, occasionally, four places. In this class we find 5, 6, and 7-place tables in which the functions are given for each 10 seconds or for each second, those tables in which the functions are given for each minute, and are interpolated for decimal parts of a minute, the 4-place tables in which the functions are given for each 10 minutes, and tables in which the differences of the successive values of the argument are tenths or hundredths of a degree.

The second class includes those tables in which the functions are given for

each minute, and the proportional parts for 6, 7, 8, 9, 10, 20, 30, 40, and 50 seconds. Gauss's or Hoüel's 5-place tables are such.

In the third class are those 5 and 6-place tables in which the functions are given for each minute, and the variations of the functions for each second, these variations being carried to two decimal places. In Olney's or Davies's 6-place tables these variations, though printed directly opposite the functions, should be placed half a line lower, since they have been so computed that they show the rate at which the function is varying at each half minute. In Schlömilch's 5-place tables the variations have been found by dividing the differences between successive tabular functions by 60.

Class I. The discussion of this class has already virtually been given in Problem I. If x be taken as 0.01, 0.02, . . . 0.99 the average probable value of $xc' + (1 - x)c$ becomes 0.810×0.25, and no appreciable change will be made if we have x equal to 0.001, 0.002, . . . 0.999. But taking into account the fact that x may be 0.0, in which case the probable error is 0.25, we find that, when x varies from 0.1 to 0.9, the average probable error is 0.31. When x varies from 0.01 to 0.99 and from 0.001 to 0.999, the average probable error is 0.32.

Class II. In Hoüel's 5-place table, the functions in the tables of proportional parts are given to the nearest integer only. If the number of seconds has two figures and does not end in a cipher, two interpolations from the table of proportional parts must be taken, and the probable error of the sum of these is $1/2 \cdot (0.25)$, or 0.35. If n denote the number of seconds and $x = n/60$, the error of the interpolated function would be (aside from the source of error just mentioned), $xc' + (1 - x)c$. Since the probable values of c' and c are each 0.25, we have the following formula for the probable error in this particular case: —

$$\text{Error} = 1/[x^2 + (1 - x)^2 + 2] \cdot (0.25).$$

When but a single number is taken from the table of proportional parts the proper formula is

$$\text{Error} = 1/[x^2 + (1 - x)^2 + 1] \cdot (0.25).$$

Substituting the values of x, which range from 1/60 to 59/60 in the preceding formulæ, we find that the average of the probable errors is 0.39, the minimum being 0.31 and the maximum 0.43.

But, if the functions in the tables of proportional parts had been given to tenths, as in Gauss's table, and the interpolated functions had been carried to 6 places, besides the quantity $xc' + (1 - x)c$ there would have been, arising from the neglect of the 7th place, a probable error of 0.025 (in units of the 5th place) when the proportional part was given directly in the table, and of 0.035 if two entrances into the interpolation-table were made. This has so slight an effect that we pay no further attention to it, and assume that the error arising from

dropping the 6th place is 0.25. The last formula given above applies to this case, and for the average probable error we find 0.32.

Class III. In Olney's or Davies's 6-place tables the difference for one second has a probable error of 0.0025 in units of the 6th place. This is multiplied by numbers ranging from 0.1 to 59.9. The average probable error of the product is thus 30×0.0025, or 0.075. Combining this with the the probable error of the mantissa to which the product is added, and remembering that 0.25 is rejected when the result is limited to 6 places, we have

$$\text{Error} = \sqrt{[(0.25)^2 + (0.075)^2 + (0.25)^2]} = 0.36.$$

The errors arising in the use of Schlömilch's 5-place table are nearly the same in magnitude as those of the tables in Class I. Let D be the apparent tabular difference and n the number of seconds. We really multiply $D/60$ (carried to hundredths) by n. But this is practically equivalent to multiplying $n/60$ (carried to hundredths) by D. Proceeding as under Case I, and including the error 0.075 which was lately mentioned, we find for the result

$$\text{Error} = \sqrt{[(0.25)^2 + (0.810)^2\,(0.25)^2 + (0.075)^2]} = 0.33.$$

Hence functions interpolated from Schlömilch's table have a less probable error than those taken from Olney's or Davies's. This accords with the conclusion reached in Problem VI.

PROBLEM XII.

Given the logarithm of the tangent of an angle, to find the probable error of the corresponding interpolated angle.

We divide the tables into the three classes given in Problem XI.

Class I. The interpolation in this class of tables is like that in tables of logarithms of natural numbers. Hence the solution of this problem is similar to that of Problem II. If but one figure is interpolated we have

$$\text{Error} = \sqrt{\left[\left(\frac{1.98}{D}\right)^2 + (0.25)^2\right]}.$$

If more than one figure is interpolated we have (see under Class I of Problem XI) instead of 1.98, 8.10 / 4, or 2.02. For this case

$$\text{Error} = \sqrt{\left[\left(\frac{0.202}{D} \cdot 10^{\prime}\right)^2 + (0.25)^2\right]}.$$

By substituting the values of D for different portions of the table, one can compute a series of values for the error.

Class II. Let us assume that the proportional parts are given to tenths, as in Gauss's table. Suppose that the angles are carried accurately to thousandths

of a minute by computing d/D. Since the neglect of the ten-thousandths would introduce a probable error of only $60'' \times 0.00025$, or $0''.015$ into the angle finally derived, and since the other errors are very much larger than this, we pay no further attention to it, but account the thousandths accurate. According to the reasoning in Problem II (remembering to use 0.810/4 instead of 0.793/4), we then have $0.202/D$ as the probable error. But, in reducing the angle to seconds, this error would be multiplied by $60''$, giving the expression $12''.12/D$. The error thus far treated of arises from the inaccuracy of the tabular logarithms. But in using the tables of proportional parts given in Gauss, further error is introduced, because the proportional parts are carried to tenths only. Now, in general, if an angle be carried to hundredths of a second, three subtractions will be made, and at each subtraction a probable error of 0.025 of a unit is introduced into the remainder. Hence the probable error of the final remainder is $0.025\sqrt{3}$. Since the change of the proportional part for $1''$ is $D/60$, the effect in seconds of the error just mentioned is $0.025\sqrt{3} \div D/60$, which equals $1.5\sqrt{3}/D = 2.6/D$. The final value of the angle being restricted to seconds, a further error, whose probable value is $0''.25$ is introduced. The three errors which we have discussed being independent of each other, the final result is

$$\text{Error} = \sqrt{\left[\left(\frac{12''.12}{D}\right)^2 + \left(\frac{2''.6}{D}\right)^2 + (0.25)^2\right]}.$$

Class III. In using Olney's or Davies's 6-place tables, the difference between the given logarithm and the tabular one has a probable error of 0.25. Denoting this difference by d, and the difference for $1''$ by D', since the latter has a probable error of 0.0025, we divide $d \pm 0.25$ by $D' \pm 0.0025$ and obtain the quotient

$$\frac{d}{D'} + \frac{0.25}{D'} \mp (0.0025)\frac{d}{D'^2} - \ldots .$$

We really adopt the value d/D'. Hence the error is $0.25/D' \mp 0.0025\, d/D'^2$. If the result is restricted to tenths of a second, as usual, another error whose probable value is 0.025 is introduced. The average value of d/D' is 30, and the formula for the average probable error becomes

$$\text{Error} = \sqrt{\left[\left(\frac{0.25}{D'}\right)^2 + \left(\frac{0.075}{D'}\right)^2 + (0.025)^2\right]},$$

or if D is taken as the difference for one minute,

$$\text{Error} = \sqrt{\left[\left(\frac{15}{D}\right)^2 + \left(\frac{4.5}{D}\right)^2 + (0.025)^2\right]}.$$

In employing Schlömilch's table, if the difference for $1''$ were carried to 4 or 5 places, and the interpolated seconds to hundredths, the result would be the same

as if d were divided by D, the quotient carried to 4 or 5 places of decimals, and then multiplied by 60″.

But in this case the error due to c and c' would be, according to Problem II,

$$\frac{d}{D^2}c' + \frac{1 - \frac{d}{D}}{D}c,$$

whose probable value has been shown to be $0.202/D$ when expressed as a decimal of a minute, or $12.12/D$, when expressed in seconds. As in Olney's table there is another error of $0.075/D'$ due to the inaccuracy of D'; when the interpolated seconds are not carried beyond units, 0.25 is the probable value of the rejected figures. Thus the final formula is (1″ being the unit)

$$\text{Error} = \sqrt{\left[\left(\frac{0.202}{D'}\right)^2 + \left(\frac{0.075}{D'}\right)^2 + (0.25)^2\right]}$$

$$= \sqrt{\left[\left(\frac{12.12}{D}\right)^2 + \left(\frac{4.5}{D}\right)^2 + (0.25)^2\right]}.$$

PROBLEM XIII.

To find the maximum errors in Problem XI.

Class I. The maximum error is 1.0 in the n^{th} place for n-place logarithms, as shown in Problem IV.

Class II. For Hoüel's table no figure is rejected from the result given by interpolation. In addition to the error caused by the incorrectness of the tabular logarithms, one is introduced depending on the fact that each of the two functions taken from the table of proportional parts may have an error of 0.5 in the fifth place, and these errors may conspire. The formula then is

$$\text{Maximum error} = xc' + (1 - x)c + 1.00$$
$$= 1.50.$$

If Gauss's table be used, the error due to cutting off the functions in the tables of proportional parts at tenths of a unit of the fifth decimal place, may amount to 0.05 + 0.05, or 0.10. Since all figures beyond the fifth place are rejected, another error of 0.5 may arise. Hence we have

$$\text{Maximum error} = xc' + (1 - x)c + 0.6$$
$$= 1.10.$$

Class III. In Olney's table an error arises from the neglect of second differences, which is often quite appreciable, but will not be treated of. The difference for 1″ may have an error of 0.005 in units of the sixth place. The product of this by 59.9 (the largest number of seconds ever employed) is 0.30. The maximum error of the tabular mantissa to which the proportional part is applied is

0.50. The rejected figures beyond the sixth place may amount to 0.50. Hence

$$\text{Maximum error} = 1.30.$$

If Schlömilch's table be used, the error in the difference for 1″, caused by terminating it at the second decimal place, may be $0.00\frac{1}{3}$ of a unit of the fifth place. This multiplied by 59 gives 0.20. This table is also subject to the error of those of Class I. Therefore

$$\text{Maximum error} = 1.20$$

PROBLEM XIV.

To find the maximum errors in Problem XII.

Class I. From Problem V we get

$$\text{Maximum error} = \frac{0.50}{\delta} \cdot 10^f + 0.50,$$

in which δ is the minimum value of D, and f denotes the number of figures interpolated. The error is given in units of the last interpolated figure. For angles taken from the logarithmic tangents of Bremiker's 6-place table the formula gives 1.7 tenths of a second, or 0″.17.

Class II. We shall not discuss Hoüel's table, because the value of the interpolated seconds frequently depends upon the computer's judgment.

In using Gauss's table let us suppose that the angles are carried to hundredths of a second, the process being to divide the difference d by the tabular difference D, and multiply the result by 60. Then by Problem II the error in seconds is

$$60'' \left[\frac{d}{D^2} c' + \frac{1 - \frac{d}{D}}{D} c - \frac{1 - \frac{d}{D}}{D^2} c\,(c' - c) + \frac{d}{D^2} c'\,(c' - c) + \ldots \right].$$

Since the first two terms of the parenthesis are much larger than the others, we take them to find the condition for a maximum. Plainly, c' must equal c. Substituting 0.50 for them, the foregoing expression becomes

$$60'' \left(\frac{0.50}{D} \right),$$

which is a maximum when D is a minimum. The least value of D in the tangent column is 25, and then the above expression = 1″.20. But the employment of Gauss's tables of proportional parts causes another error, because the functions in them are carried to tenths only. A little experimentation shows that the maximum error due to this cause, when D is near 25, is 0″.20. In restricting the angle to the nearest second a possible error of 0″.50 is made, which added to those previously given makes 1″.90. It is hardly possible that this limit of inaccuracy is ever reached in practice.

Class III. In Olney's table we assume that the difference for 1″ has *only* the error 0.005, and that the next less tabular mantissa is 0.5 in error in the 6th place. Then Problem XII gives us

$$\text{Maximum error} = \frac{0''.50}{\delta'} + \left(0''.005\,\frac{d}{\delta'}\right) \div \delta' + 0''.05,$$

in which δ' is the minimum value of D', and the interpolated seconds are terminated at tenths. Since $\delta' = 4.21$, the error is 0″.24.

In employing Schlömilch's table there are three errors, the first due to the inaccuracy of the tabular logarithms, the second to the cutting off of D' at hundredths, and the third caused by restricting the interpolated angle to the nearest second. From previous discussions we readily get

$$\text{Maximum error} = 1''.20 + \left(0''.003\tfrac{1}{3}\,\frac{d}{\delta'}\right) \div \delta' + 0''.50.$$

Assuming that $d/\delta' = 57$, and $\delta' = 0.42$, this reduces to 2″.15. In practice this error is never reached, and it is doubtful whether the limit 2″.00 can be attained.

PROBLEM XV.

To compare the theories of Problems XI and XII with the facts.

In testing the theories of these Problems, Bremiker's 6-place table was used as a specimen of those of Class I. 120 angles uniformly distributed between 5° and 65°, and carried to seconds, were chosen, and their logarithmic tangents taken from each of the five tables given below. The results were compared with those obtained from Schrön's 7-place table, which was used as a standard. The averages of the errors, in units of the n^{th} place, for n-place logarithms are given below:—

	Actual Errors.	Theoretical Errors.
Bremiker,	0.27	0.31
Hoüel,	0.39	0.39
Gauss,	0.31	0.32
Olney,	0.36	0.36
Schlömilch,	0.31	0.33

The average of the errors of the functions taken from Bremiker was unusually small.

In testing the theories of Problem XII, 120 logarithmic tangents were taken, and the corresponding angles, which ranged from 35° to 45°, were obtained. Schrön was used again as the standard. For Bremiker the average value of D was 43, and for Gauss, 26. For Olney and for Schlömilch the aver-

age values of D' were 4.30 and 0.43 respectively. The average errors are tabulated below: —

	Actual Error.	Theoretical Error.
Bremiker,	0″.050	0″.053
Gauss,	0.45	0.54
Olney,	0.063	0.066
Schlömilch,	0.50	0.56

Being dissatisfied with the discrepancies in the cases of Gauss and Schlömilch, I worked out another set of 120 for each, taking care to have the resulting seconds quite uniformly distributed throughout the minute. Gauss's error became 0″.53, but Schlömilch's was 0″.49 for forward interpolation and 0″.52 for backward interpolation. The seconds given by the interpolation backward differed by a unit from those given by the forward, in 61 of the 120 angles; this portion of the table seems to be bewitched.

REMARK.—As it may be of interest to know the different times required in using various tables, I append those for obtaining the 120 angles from their logarithmic tangents. Schrön's angles were carried to hundredths of a second, Bremiker's and Olney's to tenths, and those from the remaining tables to seconds. The degrees and minutes were written down in Schrön's angles only, the seconds alone being recorded for the other tables. A multiplication table was used to facilitate interpolation from Olney. The seconds of each angle were found twice, once by forward interpolation, and once by backward, to insure their accuracy.

	First 40.		Second 40.		Third 40.		Total.		
	m.	*sec.*	*m.*	*sec.*	*m.*	*sec.*	*h.*	*m.*	*sec.*
Schrön (7-place),	27	40	25	0	21	10	1	13	50
Bremiker (6-place),	14	0	12	15	13	45	0	40	0
Olney (6-place),	25	0	21	40	16	35	1	3	15
Gauss (5-place),	10	45	10	20	9	35	0	30	40
Schlömilch (5-place),	12	40	11	30	11	10	0	35	20
Gernerth (5-place),	5	15	4	25	4	30	0	14	10

In Gernerth's table the functions are given for each 10″.

NOTE.—Since logarithmic errors do not follow with exactness the laws of those treated of in the method of least squares, investigations based on the law assumed at the beginning of this series of articles cannot be expected to give results rigorously true; but their close approach to the truth, together with the difficulties of a rigid investigation of the subject, seem to justify the approximate method here employed.

EXPRESSION OF THE CO-EFFICIENTS OF STURM'S FUNCTIONS AS DETERMINANTS.

By Mr. Wm. E. Heal, Marion, Ind.

Let $U = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots a_{n+1} = 0$ be an algebraic equation of the n^{th} degree,

$V = b_1x^{n-1} + b_2x^{n-2} + b_3x^{n-3} + \ldots b_n$ its first derived function, and

$$f_{m-2}(x) = c_1^{(m-2)}x^{n-(m-1)} + c_2^{(m-2)}x^{n-m} + c_3^{(m-2)}x^{n-(m+1)} + \ldots c_{n-(m-2)}^{(m-2)}, \quad (1)$$

$$f_{m-1}(x) = c_1^{(m-1)}x^{n-m} + c_2^{(m-1)}x^{n-(m+1)} + c_3^{(m-1)}x^{n-(m+2)} + \ldots c_{n-(m-1)}^{(m-1)}, \quad (2)$$

$$f_m(x) = c_1^{(m)}x^{n-(m+1)} + c_2^{(m)}x^{n-(m+2)} + c_3^{(m)}x^{n-(m+3)} + \ldots c_{n-m}^{(m)}, \quad (3)$$

the $(m-2)^{th}$, $(m-1)^{th}$, and m^{th} Sturmian functions. Multiply equation (2) by an arbitrary function $A_1x + A_2$, of the first degree in x, subtract equation (1), and determine A_1, A_2 so that the two highest terms shall vanish, and we get equation (3).

$$\therefore c_p^{(m)} = c_1^{(m-2)}c_1^{(m-1)}c_{p+2}^{(m-1)} - c_{p+2}^{(m-2)}\left[c_1^{(m-1)}\right]^2 + c_2^{(m-2)}c_1^{(m-1)}c_{p+1}^{(m-1)} - c_1^{(m-2)}c_2^{(m-1)}c_{p+1}^{(m-1)}. \quad (4)$$

Equation (4) enables us to calculate $f_m(x)$ from $f_{m-2}(x)$ and $f_{m-1}(x)$.

We may express $c_p^{(m)}$ in the form of a determinant, and write

$$c_p^{(m)} = -\begin{vmatrix} c_1^{(m-2)}, & c_1^{(m-1)}, & 0 \\ c_2^{(m-2)}, & c_2^{(m-1)}, & c_1^{(m-1)} \\ c_{p+2}^{(m-2)}, & c_{p+2}^{(m-1)}, & c_{p+1}^{(m-1)} \end{vmatrix} = \begin{vmatrix} c_1^{(m-2)}, & 0, & c_1^{(m-1)} \\ c_2^{(m-2)}, & c_1^{(m-1)}, & c_2^{(m-1)} \\ c_{p+2}^{(m-2)}, & c_{p+1}^{(m-1)}, & c_{p+2}^{(m-1)} \end{vmatrix}. \quad (5)$$

The equality of the two expressions of $c_p^{(m)}$ may be verified by expanding equation (5), or we may deduce this equation at once, by observing that $c_p^{(m)}$ is the result of eliminating A_1, A_2 between the equations

$$\begin{aligned} -c_1^{(m-2)} + A_1c_1^{(m-1)} &= 0, \\ -c_2^{(m-2)} + A_1c_2^{(m-1)} + A_2c_1^{(m-1)} &= 0, \\ -c_{p+2}^{(m-2)} + A_1c_{p+2}^{(m-1)} + A_2c_{p+1}^{(m-1)} &= 0. \end{aligned}$$

In the case in which $f_{m-2}(x) = U$ and $f_{m-1}(x) = V$, we have

$$c_1^{(m-2)} = a_1, \quad c_2^{(m-2)} = a_2, \quad c_{p+2}^{(m-2)} = a_{p+2},$$

$$c_1^{(m-1)} = b_1 = na_1, \quad c_2^{(m-1)} = b_2 = (n-1)a_2, \quad c_{p+2}^{(m-1)} = b_{p+2} = (n-p-1)a_{p+2}.$$

$$\therefore c_p^{(1)} = -\begin{vmatrix} a_1, & b_1, & 0 \\ a_2, & b_2, & b_1 \\ a_{p+2}, & b_{p+2}, & b_{p+1} \end{vmatrix}. \quad (6)$$

Multiply the first column by n and subtract from the second, and we have

$$c_p^{(1)} = -\begin{vmatrix} a_1, & 0, & 0 \\ a_2, & -a_2, & b_1 \\ a_{p+2}, & -(p+1)a_{p+2}, & b_{p+1} \end{vmatrix}. \qquad (7)$$

Hence, rejecting the factor a_1, supposed positive, we may write

$$c_p^{(1)} = \begin{vmatrix} a_2, & b_1 \\ (p+1)a_{p+2}, & b_{p+1} \end{vmatrix}. \qquad (8)$$

It will, in general, be most convenient to deduce each of Sturm's functions from the two preceding by the aid of equations (3) and (5), but it may be desirable to express each function in terms of U and V; this may be done as follows: —

$$\begin{aligned} f_1(x) &= Q_1 V - U, \\ f_2(x) &= Q_2 f_1(x) - V \\ &= Q_1 Q_2 V - Q_2 U - V \\ &= (Q_1 Q_2 - 1) V - Q_2 U, \\ f_3(x) &= Q_3 f_2(x) - f_1(x) \\ &= Q_3 (Q_1 Q_2 - 1) V - Q_2 Q_3 V - Q_1 V + U \\ &= (Q_1 Q_2 Q_3 - Q_1 - Q_3) V - (Q_2 Q_3 - 1) U, \end{aligned}$$

where Q_1, Q_2, Q_3 are functions of the first degree in x.

In general
$$f_m(x) = P_m V - R_m U, \qquad (9)$$

where P_m, R_m are functions of x of degree m, $m-1$ respectively.

If
$$\begin{aligned} P_m &= A_1 x^m + A_2 x^{m-1} + A_3 x^{m-2} + \ldots A_{m+1}, \\ R_m &= B_1 x^{m-1} + B_2 x^{m-2} + B_3 x^{m-3} + \ldots B_m, \end{aligned}$$

we seem to have $2m+1$ unknown quantities, but really have one less, since any one of the quantities A, B may be made $= 1$ by division; and, since we have to make the first $2m$ terms of equation (9) vanish, we have the right number of equations to enable us to determine the values of $A_1, A_2, A_3, \ldots A_{m+1}, B_1, B_2, B_3, \ldots B_m$. Expanding equation (9), we see that $c_p^{(m)}$ is the result of eliminating $A_1, A_2, A_3, \ldots A_{m+1}, B_1, B_2, B_3, \ldots B_m$ between the equations

$$\begin{aligned} &-B_1 a_1 && +A_1 b_1 \\ &-B_1 a_2 - B_2 a_1 && +A_1 b_2 + A_2 b_1 \\ &-B_1 a_3 - B_2 a_2 - B_3 a_1 && +A_1 b_3 + A_2 b_2 + A_3 b_1 \\ &-B_1 a_4 - B_2 a_3 - B_3 a_2 - B_4 a_1 && +A_1 b_4 + A_2 b_3 + A_3 b_2 + A_4 b_1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &-B_1 a_m - B_2 a_{m-1} - B_3 a_{m-2} - B_4 a_{m-3} \ldots - B_m a_1 \\ &\qquad + A_1 b_m + A_2 b_{m-1} \ldots + A_m b_1 \end{aligned}$$

$$
\begin{aligned}
&- B_1 a_{m+1} - B_2 a_m - B_3 a_{m-1} - B_4 a_{m-2} \cdot \cdot \cdot - B_m a_2 \\
&\qquad + A_1 b_{m+1} + A_2 b_m \cdot \cdot \cdot + A_m b_2 - A_{m+1} b_1 \\
&- B_1 a_{m+2} - B_2 a_{m+1} - B_3 a_m - B_4 a_{m-1} \cdot \cdot \cdot - B_m a_3 \\
&\qquad + A_1 b_{m+2} + A_2 b_{m+1} \cdot \cdot \cdot + A_m b_3 - A_{m+1} b_2 \\
&\cdot \cdot \cdot \

$$\therefore\ c_p^{(m)} = \begin{vmatrix}
-a_1, & 0, & 0, & 0, & \dots & 0, & b_1, & 0, & \dots & 0, & 0 \\
-a_2, & -a_1, & 0, & 0, & \dots & 0, & b_2, & b_1, & \dots & 0, & 0 \\
-a_3, & -a_2, & -a_1, & 0, & \dots & 0, & b_3, & b_2, & \dots & 0, & 0 \\
-a_4, & -a_3, & -a_2, & -a_1, & \dots & 0, & b_4, & b_3, & \dots & 0, & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
-a_m, & -a_{m-1}, & -a_{m-2}, & -a_{m-3}, & \dots & -a_1, & b_m, & b_{m-1}, & \dots & b_1, & 0 \\
-a_{m+1}, & -a_m, & -a_{m-1}, & -a_{m-2}, & \dots & -a_2, & b_{m+1}, & b_m, & \dots & b_2, & -b_1 \\
-a_{m+2}, & -a_{m+1}, & -a_m, & -a_{m-1}, & \dots & -a_3, & b_{m+2}, & b_{m+1}, & \dots & b_3, & -b_2 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
-a_{2m-1}, & -a_{2m-2}, & -a_{2m-3}, & -a_{2m-4}, & \dots & -a_m, & b_{2m-1}, & b_{2m-2}, & \dots & b_m, & -b_{m-1} \\
-a_{2m}, & -a_{2m-1}, & -a_{2m-2}, & -a_{2m-3}, & \dots & -a_{m+1}, & b_{2m}, & b_{2m-1}, & \dots & b_{m+1}, & -b_m \\
-a_{2m+p}, & -a_{2m+p-1}, & -a_{2m+p-2}, & -a_{2m+p-3}, & \dots & -a_{m+p+1}, & b_{2m+p}, & b_{2m+p-1}, & \dots & b_{m+p+1}, & -b_{m+p}
\end{vmatrix} . \quad (10)$$

Add n times the first column to the $(m+1)^{th}$, n times the second column to the $(m+2)^{th}$, and so on, observing that $a_k = (n-k+1)\,b_k$, reject the factor a_1, and we have

$$c_p^{(m)} = -\begin{vmatrix}
a_1, & 0, & 0, & \dots & 0, & a_2, & 0, & \dots & 0, & 0 \\
a_2, & a_1, & 0, & \dots & 0, & 2a_3, & a_2, & \dots & 0, & 0 \\
a_3, & a_2, & a_1, & \dots & 0, & 3a_4, & 2a_3, & \dots & 0, & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_{m-1}, & a_{m-2}, & a_{m-3}, & \dots & a_1, & (m-1)\,a_m, & (m-2)\,a_{m-1}, & \dots & 0, & 0 \\
a_m, & a_{m-1}, & a_{m-2}, & \dots & a_2, & ma_{m+1}, & (m-1)\,a_m, & \dots & a_2, & b_1 \\
a_{m+1}, & a_m, & a_{m-1}, & \dots & a_3, & (m+1)\,a_{m+2}, & ma_{m+1}, & \dots & 2a_3, & b_2 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_{2m-2}, & a_{2m-3}, & a_{2m-4}, & \dots & a_m, & (2m-2)\,a_{2m-1}, & (2m-3)\,a_{2m-2}, & \dots & (m-1)\,a_m, & b_{m-1} \\
a_{2m-1}, & a_{2m-2}, & a_{2m-3}, & \dots & a_{m+1}, & (2m-1)\,a_{2m}, & (2m-2)\,a_{2m-1}, & \dots & ma_{m+1}, & b_m \\
a_{2m+p-1}, & a_{2m+p-2}, & a_{2m+p-3}, & \dots & a_{m+p+1}, & (2m+p-1)\,a_{2m+p}, & (2m+p-2)\,a_{2m+p-1}, & \dots & (m+p)\,a_{m+p+1}, & b_{m+p}
\end{vmatrix} . \quad (11)$$

EXAMPLE :— Let $U = x^4 - 8x^3 + 14x^2 + 4x - 8 = 0,$
$V = x^3 - 6x^2 + 7x + 1.$

By equation (8)

$$c_1^{(1)} = \begin{vmatrix} -8, & 1 \\ 28, & -6 \end{vmatrix} = 20,$$

$$c_2^{(1)} = \begin{vmatrix} -8, & 1 \\ 12, & 7 \end{vmatrix} = -68,$$

$$c_3^{(1)} = \begin{vmatrix} -8, & 1 \\ -32, & 1 \end{vmatrix} = 24;$$

$$\therefore f_1(x) = 5x^2 - 17x + 6.$$

By equation (5)

$$c_1^{(2)} = \begin{vmatrix} 1, & 0, & 5 \\ -6, & 5, & -17 \\ 7, & -17, & 6 \end{vmatrix} = 76,$$

$$c_2^{(2)} = \begin{vmatrix} 1, & 0, & 5 \\ -6, & 5, & -17 \\ 1, & 6, & 0 \end{vmatrix} = -103;$$

$$\therefore f_2(x) = 76x - 103.$$

$$f_3(x) = c_1^{(3)} = \begin{vmatrix} 5, & 0, & 76 \\ -17, & 76, & -103 \\ 6, & -103, & 0 \end{vmatrix} = 45375.$$

To calculate $f_2(x)$ and $f_3(x)$ by equation (11) we have

$$c_1^{(2)} = -\begin{vmatrix} 1, & -8, & 0, & 0 \\ -8, & 28, & -8, & 1 \\ 14, & 12, & 28, & -6 \\ 4, & -32, & 12, & 7 \end{vmatrix} = 16 \times 76,$$

$$c_2^{(2)} = -\begin{vmatrix} 1, & -8, & 0, & 0 \\ -8, & 28, & -8, & 1 \\ 14, & 12, & 28, & -6 \\ -8, & 0, & -32, & 1 \end{vmatrix} = -16 \times 103,$$

$$c_1^{(3)} = -\begin{vmatrix} 1, & 0, & -8, & 0, & 0, & 0 \\ -8, & 1, & 28, & -8, & 0, & 0 \\ 14, & -8, & 12, & 28, & -8, & 1 \\ 4, & 14, & -32, & 12, & 28, & -6 \\ -8, & 4, & 0, & -32, & 12, & 7 \\ 0, & -8, & 0, & 0, & -32, & 1 \end{vmatrix} = 64 \times 1815.$$

SOLUTIONS OF EXERCISES.

64

A FLY-WHEEL weighing 12000 pounds with a rim of 10 feet diameter is driven by a piston pressure of 3000 pounds acting through a stroke of 3 feet. If the wheel be treated as concentrated in its rim, the weight of all other parts of the mechanism being neglected and the connecting-rod being treated as infinite in length, how many turns will the wheel make in 10 seconds?

[*DeVolson Wood.*]

SOLUTION.

Let ϑ = angle between crank and connecting-rod,
a = length of crank,
F = piston pressure,
I = moment of inertia of wheel.

Then
$$\frac{d^2\vartheta}{dt^2} = m \sin \vartheta, \tag{1}$$

where
$$m = Fa / I.$$

For the first stroke, let a = initial value of ϑ. Then
$$\left(\frac{d\vartheta}{dt}\right)^2 = 2m(\cos a - \cos \vartheta), \tag{2}$$

$$\sqrt{(2m)} \cdot t = \int_a^{\vartheta} \frac{d\vartheta}{\sqrt{(\cos a - \cos \vartheta)}}.$$

Let $\sin \varphi = \cos \frac{1}{2}\vartheta / \cos \frac{1}{2}a$. Then
$$\sqrt{(m)} \cdot t = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\Delta\varphi} = K - F(k, \varphi), \tag{3}$$

where $k = \cos \frac{1}{2}a$.

For any succeeding stroke, let ω = initial angular velocity, and 0 = initial value of ϑ; then
$$\left(\frac{d\vartheta}{dt}\right)^2 = \omega^2 + 2m(1 - \cos \vartheta). \tag{4}$$

Let $\vartheta = \pi - 2\varphi$; then
$$\tfrac{1}{2}\sqrt{(\omega^2 + 4m)} \cdot t = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\Delta\varphi} = K - F(k, \varphi), \tag{5}$$

where $k = \sqrt{\frac{4m}{\omega^2 + 4m}}$.

For the whole time of any stroke, $F(k, \varphi)$ in (3) or (5) becomes 0, and we have

$$h_n t_n = K_n, \tag{6}$$

the subscript denoting the number of the stroke; or

$$\log t_n = \log K_n - \log h_n. \tag{7}$$

Here $h_1 = \sqrt{m}$, $h_2 = \frac{1}{2}\sqrt{(\omega_2^2 + 4m)}$, $h_3 = \frac{1}{2}\sqrt{(\omega_3^2 + 4m)}$, etc.

$$k_1 = \cos\frac{a}{2}, \quad k_2 = \sqrt{\frac{4m}{\omega_2^2 + 4m}}, \qquad k_3 = \sqrt{\frac{4m}{\omega_3^2 + 4m}}, \text{ etc.}$$

From (2) and (4), $\omega_2^2 = 2m(\cos a + 1)$, $\omega_3^2 = 2m(\cos a + 3)$, etc. Assume $a = \frac{1}{2}\pi$, then $\omega_2^2 = 2m$, $\omega_3^2 = 6m$, $\omega_4^2 = 10m$, etc.

In the problem given, $F = 3000$, $a = \frac{3}{2}$, $I = \frac{1}{3}\frac{2}{2}\frac{0}{2}\frac{0}{2} \times 25$; $\therefore\ m = 0.483$. Hence we may find

$\log h_1 = 9.8420$	$\log k_1 = 9.8495$	$\log K_1 = 0.2681$
$\log h_2 = 9.9300$	$\log k_2 = 9.9120$	$\log K_2 = 0.3073$
$\log h_3 = 0.0409$	$\log k_3 = 9.8010$	$\log K_3 = 0.2498$
$\log h_4 = 0.1140$	$\log k_4 = 9.7280$	$\log K_4 = 0.2319$
$\log h_5 = 0.1686$	$\log k_5 = 9.6734$	$\log K_5 = 0.2230$
$\log h_6 = 0.2122$	$\log k_6 = 9.6298$	$\log K_6 = 0.2177$

The values of $K_1, \ldots K_6$ were found from the table, Hoüel's *Calcul Infinitésimal*, Vol. IV, p. 302.

Substituting in (7), we may find

$$t_1 = 2.668, \quad t_2 = 2.384, \quad t_3 = 1.618, \quad t_4 = 1.312, \quad t_5 = 1.133.$$

The whole time of the first five strokes is 9.115 seconds.

To find the angle described during the remaining 0.885 seconds we have from (5),

$$\begin{aligned} \varphi &= \text{am}\,[K_6 - \tfrac{1}{2}\sqrt{(\omega_6^2 + 4m)}\,.\,t_6], \quad \text{mod. } k_6 \\ &= \text{am } 0.208, \quad \text{mod. } 0.4264. \end{aligned}$$

By interpolation in the table (*Calcul Infinitésimal*, Vol. IV, p. 303), we find $\varphi = 0.132\pi/2$; $\therefore\ \vartheta = \pi - 2\varphi = 0.868\pi$. The whole angle described in 10 seconds is therefore $\frac{1}{2}\pi + 4\pi + 0.868\pi = 5.368\pi$, and the number of revolutions is 2.684. [*L. M. Hoskins.*]

66

SHOW that the difference between the radii of the inscribed and circumscribed circles of a triangle is a minimum when the triangle is equilateral.

[*R. D. Bohannan.*]

SOLUTION.

If r and R be the radii of the inscribed and circumscribed circles of the triangle ABC, considering R as constant, we shall have

$$1 - r/R = 2 - \cos A - \cos B - \cos C = \text{min.}$$

Considering C as constant,

$$dA = -dB,$$
$$-\sin A + \sin B = 0,$$

and similarly with regard to B, C and C, A;

$$\therefore A = B = C. \qquad [\textit{Ormond Stone.}]$$

67

A WINS from B on an average 8 games of chess out of 11. What is the value of A's chance in a match for $100 to be given to the player who shall first score three games; drawn games not to be counted? [*L. G. Barbour.*]

SOLUTION.

Let p be the probability that A will win the match; then $100p$ will be the value of A's chance. Denote by p_1, p_2, p_3 the respective probabilities that A will win the third and the two preceding games, the fourth and two of the preceding games, the fifth and two of the preceding games; then will

$$p = p_1 + 3p_2 + 6p_3;$$

for A can evidently win two of three games in three ways, and two of four games in six ways. Now

$$p_1 = (\tfrac{8}{11})^3,$$
$$p_2 = (\tfrac{8}{11})^3(\tfrac{3}{11}),$$
$$p_3 = (\tfrac{8}{11})^3(\tfrac{3}{11})^2;$$

therefore $$p = p_1 + 3p_2 + 6p_3 = \tfrac{1402888}{1610511},$$

and $100p$, the value of A's chance, is

$$\$14028800 \div 161051 = \$87.10 +. \qquad [\textit{A. B. Evans.}]$$

68

FROM a point P in the plane of a parabola normals are drawn to the curve. Find the locus of P when the feet of the normals are the corners of a right triangle. [*W. M. Thornton.*]

SOLUTION.

Let $y^2 = 4ax$ be the equation of the parabola, x_1, y_1; x_2, y_2; x_3, y_3 be the co-ordinates of the feet of the normals, and x, y the co-ordinates of P. Then

$$\frac{y_2 - y_1}{x_2 - x_1} \cdot \frac{y_2 - y_3}{x_2 - x_3} = -1,$$

or $$\frac{y_2 - y_1}{y_2^2 - y_1^2} \cdot \frac{y_2 - y_3}{y_2^2 - y_3^2} = -\frac{1}{16a^2},$$

or $$(y_1 + y_2)(y_2 + y_3) = -16a^2. \qquad (1)$$

Also y_1, y_2, and y_3 are the roots of the cubic in y',

$$(y - y') + \frac{y'}{2a}\left(x - \frac{y'^2}{4a}\right) = 0,$$

or $$y'^3 + (8a^2 - 4ax)y' - 8a^2y = 0;$$

$$\therefore y_1 + y_2 + y_3 = 0, \qquad (2)$$

$$y_1y_2 + y_3(y_1 + y_2) = 8a^2 - 4ax, \qquad (3)$$

$$y_1y_2y_3 = 8a^2y. \qquad (4)$$

From (1), (2), (3), and (4),

$$y_1(y_1 + y_2) = 16a^2, \qquad (5)$$

$$y_1y_2 - (y_1 + y_2)^2 = 8a^2 - 4ax, \qquad (6)$$

$$y_1y_2(y_1 + y_2) = -8a^2y. \qquad (7)$$

From (5) and (7), $$y_2 = -\tfrac{1}{2}y. \qquad (8)$$

From (5) and (6), $$y_2^2 = 4ax - 24a^2. \qquad (9)$$

From (8) and (9) we have $y^2 = 16a(x - 6a)$.

Therefore the required locus is a parabola.

REMARK.—The locus passes through the points of intersection of the original parabola and its evolute, and touches the latter at the same points.

[*R. H. Graves.*]

69

A STRAIGHT LINE is tangent to the parabola $8x - y^2$ and to the circle $x^2 + y^2 - 4$. Required the angle which this line makes with the x-axis.

[*O. Root.*]

SOLUTION.

If m denotes the tangent of the required angle, the line will have for its equation

$$y = mx + \frac{2}{m},$$

or $$y = mx + 2\sqrt{(1 + m^2)}.$$

That these may be identical we must have

$$m^4 + m^2 = 1;$$

$$\therefore m^2 = \tfrac{1}{2}(\sqrt{5} - 1).$$

There are accordingly two solutions, represented by the equations

$$y = \pm 1/[\tfrac{1}{2}(1.5 - 1)] \cdot [x + 1/5 + 1],$$

and the required angle is

$$\pm \tan^{-1} 1/[\tfrac{1}{2}(1/5 - 1)],$$

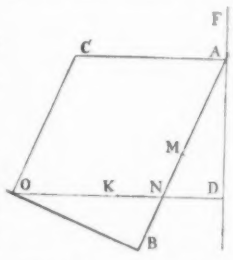
or $\pm 38^\circ 10'$ nearly. [*T. U. Taylor.*]

70

A RIGHT ANGLE moves so that a given point in one side, distant c from the vertex, lies in a fixed axis, while the other side passes through a fixed point, distant c from this axis. Find the locus of the instantaneous centres or centrodes of the motion. [*R. H. Graves.*]

SOLUTION.

Let B be the moving right angle, A the given point lying in the fixed axis DF, while O is a fixed point through which one side constantly passes. Also let $AB = OD$. Draw AC and OC at right angles to AD and OB. Their intersection C is the instantaneous centre.



Evidently $ON = AN$; $\therefore$ $CO = CA$; $\therefore$ locus of C in the fixed plane is a parabola whose focus is O and directrix is FD. Also, in the moving plane, the locus of C is an equal parabola whose focus is A and directrix is OB. M and K, the middle points of AB and OD and the vertices of the parabolas, coincide when AB coincides with OD.

Newton showed that M generates a cissoid. Also, A moves in a straight line. The cissoid and right line are known to be the loci of the vertex and focus of a parabola rolling on an equal parabola, corresponding points being in contact. [*R. H. Graves.*]

[Prof. Bohannan solves in the same way. Dr. Hendricks refers us to a solution in the *Analyst*, V, 55.]

71

A LINE of unit length is bent to the arc of a circle such that the area of the segment it determines is a maximum. Find the radius of the circle and the form and area of the segment. [*O. Root, Jr.*]

SOLUTION.

Assuming that the unit length is not greater than one circumference, the area of the segment is

$$\frac{r}{2} - \frac{r^2}{2} \sin \frac{1}{r},$$

the angle being expressed in radial measure. For a maximum or minimum value,

$$\cos\frac{1}{2r}\left(\cos\frac{1}{2r} - 2r\sin\frac{1}{2r}\right) = 0,$$

i. e. $\cos\frac{1}{2r} = 0$, or $\tan\frac{1}{2r} = \frac{1}{2r}$.

The second relation is only true for a zero-angle, and need not be considered. $\cos\frac{1}{2r} = 0$ gives, under our assumption,

$$\frac{1}{2r} = \frac{\pi}{2}; \quad \therefore r = \frac{1}{\pi}.$$

The variable factor of the second differential co-efficient of the given function is

$$-2\sin\frac{1}{r} + \frac{2}{r}\cos\frac{1}{r} + \frac{1}{r^2}\sin\frac{1}{r}.$$

Since $1/r = \pi$, this expression is negative; therefore the above value of r corresponds to a maximum.

Thus the segment is a semi-circle, and its area, in terms of the square on the given unit (a semi-circumference), is $r/2$ or $1/2\pi$.

[*R. D. Bohannan.*]

EXERCISES.

79

CONSTRUCT a square; given one vertex and two circles on which the extremities of the opposite diagonal lie. [*R. D. Bohannan.*]

80

IN exercise 4, what is the probability that the random circle exceeds the average circle. [*Artemas Martin.*]

81

FIND the equation to the curve whose ordinates represent the areas of the triangles in exercise 68. [*R. H. Graves.*]

82

THE major axes of two similar and equal concentric ellipses intersect at right angles, and the area common to the two curves is half that of either ellipse. Find the excentricity. [*Ormond Stone.*]

83

THE middle point of the segment AB is M. Find the locus of P when PM is a mean proportional between PA and PB. [*L. G. Carpenter.*]

84

FIND the thinnest frustum that can be cut from a given right circular cone by a plane parallel to the base, subject to the condition that it may be laid on its slant surface on a horizontal plane without toppling over.

[*W. M. Thornton.*]

SELECTED.

85

FIND the locus of the points from which the sum of the squares of the normals to the parabola $y^2 = 2px$ is constant.

86

SHOW that the lines of curvature of the hyperbolic paraboloid contain the points on the surface which are equidistant from the two principal rectilinear generatrices.

87

FIND the angle between the axes that $x^2 + xy + y^2 = 0$ may be at right angles.

88

AN ellipse referred to equi-conjugate diameters inclined at ω, has for equation $x^2 + y^2 = c^2$. Find the equation with reference to the same axes to the locus of the intersection point of orthogonal tangents.

89

SOLVE the cubic $9x^3 + 9x = 2$.

90

SHOW that if x be acute, one value of $\log (1 + \cos 2x + i \sin 2x)$ is $ix + \log (2 \cos x)$.

91

AN equilateral triangle, side $2a$, slides in a plane with one angular point on each of two rectangular axes. Find the locus of the third vertex.

92

From the point of contract of two equal circles α, β points A, B move on their circumferences with equal velocities in opposite directions. Find the motion of B relative to A.

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TABLES FOR CALCULATING THE CUBIC CONTENTS OF EXCAVATIONS AND EMBANKMENTS. By John R. Hudson. New York: John Wiley & Sons.

ERRATA.

Vol. II. p. 60, first and last formula, *for* r *read* r_1.

Vol. II. p. 61, first formula should be

$$b = \frac{1-x}{2x} \cdot \frac{r_1}{1 - \frac{r_1}{2ax}}$$

$$= \frac{1-x}{2x} r_1, \text{ when the orbit of the comet is a parabola.}$$

Vol. II. p. 62, formula for cubic equation, *for* $a_0 a_1^{\frac{1}{2}}$, *read* $a_0^{\frac{1}{2}} a_1$, and *for* tan φ *read* tan $\frac{1}{2}\varphi$.